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Inequalities for the Polar Derivative of a Polynomial

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1. Introduction and Statement of Results

If $P(z)$ is a polynomial of degree n such that $\text{Max}_{|z|=1} |P(z)| = 1$, then

$$\underset{|z|=1}{\text{Max}} |P'(z)| \leqslant n \tag{1}$$

and

$$\underset{|z|=R>1}{\text{Max}} |P(z)| \leqslant R^n. \tag{2}$$

Inequality (1) is an immediate consequence of S. Bernstein's theorem on the derivative of a trigonometric polynomial (for reference see [9]). Inequality (2) is a simple deduction from the Maximum Modulus Principle (see [8, p. 346] or [7, p. 158, Problem III, p. 269]).

If we restrict ourselves to the class of polynomials having no zero in the disk $|z| < 1$, then the inequality (1) can be sharpened. In fact, P. Erdös conjectured and later P. D. Lax [4] proved that if $P(z) \neq 0$ in $|z| < 1$, then (1) can be replaced by

$$\underset{|z|=1}{\text{Max}} |P'(z)| \leqslant \frac{n}{2}. \tag{3}$$

Ankeny and Rivlin [3] used (3) to prove that if $|P(z)| \leqslant 1$ for $|z| = 1$ and $P(z) \neq 0$ in $|z| < 1$, then

$$\underset{|z|=R>1}{\text{Max}} |P(z)| \leqslant \frac{R^n + 1}{2}. \tag{4}$$

In both (3), (4) equality holds for $P(z) = \alpha + \beta z^n$ where $|\alpha| = |\beta| = \frac{1}{2}$.

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Let $D_\alpha P(z)$ denote the polar differentiation of the polynomial $P(z)$ of degree n with respect to the point α, then

$$D_\alpha P(z) = nP(z) + (\alpha - z) P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative. Now corresponding to a given nth degree polynomial $P(z)$, we construct a sequence of polar derivatives

$$D_{\alpha_1} P(z) = nP(z) + (\alpha_1 - z) P'(z),$$

$$D_{\alpha_1} \cdots D_{\alpha_k} P(z) = (n-k+1) D_{\alpha_1} \cdots D_{\alpha_{k-1}} P(z) + (\alpha_k - z)(D_{\alpha_1} \cdots D_{\alpha_{k-1}} P(z))', \quad k = 2, 3, ..., n.$$

The points $\alpha_1, \alpha_2, ..., \alpha_k$, $k = 1, 2, ..., n$, may be equal or unequal. Like the kth ordinary derivative $P^{(k)}(z)$ of $P(z)$, the kth polar derivative $D_{\alpha_1} \cdots D_{\alpha_k} P(z)$ of $P(z)$ is a polynomial of degree $n-k$.

In the present paper we shall obtain several sharp inequalities concerning the maximum modulus of the polar derivative of a polynomial $P(z)$. We shall first extend (3) and (4) to the polar derivatives and thereby obtain a compact generalization of these results as well. We prove

THEOREM 1. *If $P(z)$ is a polynomial of degree n such that* $\operatorname{Max}_{|z|=1} |P(z)| = 1$ *and $P(z)$ has no zeros in the disk $|z| < 1$, then for $|z| \geq 1$*

$$|D_{\alpha_1} \cdots D_{\alpha_k} P(z)| \leq \frac{n(n-1)\cdots(n-k+1)}{2} \{|\alpha_1 \cdots \alpha_k z^{n-k}| + 1\}, \tag{5}$$

where $|\alpha_i| \geq 1$ for all $i = 1, 2, ..., k$. The result is best possible and equality in (5) *holds for the polynomial $P(z) = (z^n + 1)/2$.*

The following corollary, which immediately follows from Theorem 1, is a compact generalization of (3) and (4).

COROLLARY 1. *If $P(z)$ is a polynomial of degree n such that* $\operatorname{Max}_{|z|=1} |P(z)| = 1$ *and $P(z)$ has no zeros in the disk $|z| < 1$, then for every real or complex number α, with $|\alpha| \geq 1$, we have*

$$|D_\alpha P(z)| \leq \frac{n}{2} (|\alpha z^{n-1}| + 1), \qquad \text{for} \quad |z| \geq 1. \tag{6}$$

The result is best possible and equality in (6) *holds for the polynomial $P(z) = az^n + b$ where $|a| = |b| = \frac{1}{2}$ and $\alpha \geq 1$.*

Remark 1. Dividing the two sides of (6) by α, letting $\alpha \to \infty$, and noting that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z),$$

we get

$$|P'(z)| \leqslant \frac{n}{2} |z|^{n-1}, \qquad \text{for} \quad |z| \geqslant 1,$$

which in particular gives (3).

Next taking $z = \alpha$ in (6) and noting that

$$\{D_\alpha P(z)\}_{z=\alpha} = nP(\alpha),$$

we obtain

$$n\,|P(\alpha)| \leqslant \frac{n}{2} (|\alpha|^n + 1)$$

for every α with $|\alpha| \geqslant 1$. This is clearly equivalent to (4).

If we write

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0,$$

then

$$\begin{aligned} D_\alpha P(z) &= nP(z) + (\alpha - z)\, P'(z) \\ &= (n\alpha a_n + a_{n-1}) z^{n-1} + ((n-1)\,\alpha a_{n-1} + 2a_{n-2}) z^{n-2} \\ &\quad + \cdots + (2\alpha a_2 + (n-1) a_1) z + (\alpha a_1 + n a_0). \end{aligned}$$

If the polynomial $P(z)$ has no zeros in the disk $|z| < 1$, then by Theorem 1, with $k = 1$, it follows that

$$\begin{aligned} &|(n\alpha a_n + a_{n-1}) z^{n-1} + \cdots + (2\alpha a_2 + (n-1) a_1) z + (\alpha a_1 + n a_0)| \\ &\quad \leqslant \frac{n}{2} \{|\alpha|\, |z|^{n-1} + 1\} \operatorname*{Max}_{|z|=1} |P(z)|, \end{aligned}$$

for $|z| \geqslant 1$ and $|\alpha| \geqslant 1$. Dividing the two sides of this inequality by $|z|^{n-1}$ and letting $|z| \to \infty$, we easily obtain

$$|n\alpha a_n + a_{n-1}| \leqslant \frac{n}{2} |\alpha| \operatorname*{Max}_{|z|=1} |P(z)|$$

for every α with $|\alpha| \geqslant 1$. Choosing an argument of α suitably, we immediately get the following interesting result.

Corollary 2. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n which does not vanish in the disk $|z| < 1$, then*

$$n\,|a_n| + |a_{n-1}| \leqslant \frac{n}{2} \operatorname*{Max}_{|z|=1} |P(z)|.$$

If $P(z)$ is a self-inversive polynomial, that is if $P(z) \equiv Q(z)$ where $Q(z) = z^n \overline{P(1/\bar{z})}$ and $\operatorname{Max}_{|z|=1} |P(z)| = 1$, then [2, 10]

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant \frac{n}{2}. \tag{7}$$

We shall extend (7) to the polar derivatives of $P(z)$ by establishing the following result.

THEOREM 2. *If $P(z)$ is a self-inverse polynomial of degree n such that $\operatorname{Max}_{|z|=1} |P(z)| = 1$ and $\alpha_1, \alpha_2, ..., \alpha_k$ are real or complex numbers, then*

$$|D_{\alpha_1} \cdots D_{\alpha_k} P(z)| \leqslant \frac{n(n-1)\cdots(n-k+1)}{2} \times \{|\alpha_1 \cdots \alpha_k z^{n-k}| + 1\}, \tag{8}$$

for $|z| \geqslant 1$ and $|\alpha_i| \geqslant 1$, $i = 1, 2, ..., k$. The inequality (8) also holds for $|z| \leqslant 1$ and $|\alpha_i| \leqslant 1$, $i = 1, 2, ..., k$. The result is best possible and equality in (8) holds for the polynomial $P(z) = (z^n + 1)/2$.

COROLLARY 3. *If $P(z)$ is a self-inversive polynomial of degree n such that $\operatorname{Max}_{|z|=1} |P(z)| = 1$, then for every real or complex number α*

$$|D_\alpha P(z)| \leqslant \frac{n}{2} \quad (|\alpha z^{n-1}| + 1), \qquad \text{for} \quad |z| \geqslant 1 \quad \text{and} \quad |\alpha| \geqslant 1. \tag{9}$$

The inequality (9) also holds for $|z| \leqslant 1$ and $|\alpha| \leqslant 1$. The result is best possible and equality in (9) holds for the polynomial $P(z) = (z^n + 1)/2$.

If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

is a self-inversive polynomial of degree n, then from the second part of Corollary 3 we have

$$|D_\alpha P(z)|_{z=0} \leqslant \frac{n}{2} \operatorname*{Max}_{|z|=1} |P(z)|, \qquad \text{for} \quad |\alpha| \leqslant 1.$$

This gives

$$|nP(0) + \alpha P'(0)| \leqslant \frac{n}{2} \operatorname*{Max}_{|z|=1} |P(z)|, \qquad \text{for} \quad |\alpha| = 1.$$

Equivalently,

$$|na_0 + \alpha a_1| \leqslant \frac{n}{2} \operatorname*{Max}_{|z|=1} |P(z)|, \qquad \text{for} \quad |\alpha| = 1.$$

Choosing now an argument of α suitably we obtain the following result.

COROLLARY 4. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a self-inversive polynomial of degree n, then*

$$n\,|a_0| + |a_1| \leqslant \frac{n}{2} \operatorname*{Max}_{|z|=1} |P(z)|.$$

As a generalization of (3), it was shown by Malik [5] that if $P(z)$ is a polynomial of degree n such that $|P(z)| \leqslant 1$ for $|z| \leqslant 1$ and $P(z)$ has no zeros in the disk $|z| < k$ where $k \geqslant 1$, then

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k}. \tag{10}$$

Here we shall extend (10) to the polar derivatives of $P(z)$ and thereby give an independent proof of (10) as well. We prove

THEOREM 3. *If $P(z)$ is a polynomial of degree n such that $\operatorname{Max}_{|z|=1} |P(z)| = 1$ and $P(z)$ has no zeros in the disk $|z| < k$ where $k \geqslant 1$, then for every real or complex number β, with $|\beta| \geqslant 1$,*

$$\operatorname*{Max}_{|z|=1} |D_\beta P(z)| \leqslant n \left\{ \frac{k + |\beta|}{1+k} \right\}. \tag{11}$$

The result is best possible and equality in (11) holds for the polynomial $P(z) = (z+k)^n/(1+k)^n$ with a real number $\beta \geqslant 1$ and $k \geqslant 1$.

2. LEMMAS

For the proofs of these theorems we need the following lemmas.

LEMMA 1. *If all the zeros of an nth degree polynomial $P(z)$ lie in a circular region C and if none of the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ lie in the region C, then each of the polar derivatives*

$$D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z), \qquad k = 1, 2, \ldots, n-1,$$

has all of its zeros in C.

This follows by repeated application of Laguerre's theorem (see [1] or [6, p. 52]).

LEMMA 2. *If* $P(z)$ *is a polynomial of degree* n *such that* $\operatorname{Max}_{|z|=1}|P(z)|=1$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_k$, $k \leqslant n-1$, *are complex numbers with* $|\alpha_i| \geqslant 1$ *for all* $i=1, 2, \ldots, k$, *then for* $|z| \geqslant 1$,

$$|D_{\alpha_1} \cdots D_{\alpha_k} P(z)| + |D_{\alpha_1} \cdots D_{\alpha_k} Q(z)| \leqslant n(n-1) \cdots (n-k+1)\{|\alpha_1 \cdots \alpha_k|\,|z|^{n-k}+1\}, \tag{12}$$

where $Q(z)=z^n\overline{P(1/\bar{z})}$.

Proof of Lemma 2. For any complex number β, $|\beta|>1$, the polynomial $F(z)=P(z)-\beta z^n$ has all its zeros in $|z|<1$. Therefore, the polynomial

$$G(z)=z^n\overline{F(1/\bar{z})}=z^n\overline{P(1/\bar{z})}-\bar{\beta}=Q(z)-\bar{\beta}$$

has all its zeros in $|z|>1$ and

$$|G(z)| \leqslant |F(z)|, \qquad \text{for} \quad |z| \geqslant 1. \tag{13}$$

It follows by Rouche's theorem that for every α, $|\alpha|>1$, the polynomial $G(z)-\alpha F(z)$ has all its zeros in $|z|<1$, which implies by Lemma 1 that for complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, $|\alpha_i| \geqslant 1$, $1 \leqslant i \leqslant k$, the polynomial

$$D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k}(G(z)-\alpha F(z))$$

has all its zeros in $|z|<1$, which is equivalent to

$$|D_{\alpha_1} \cdots D_{\alpha_k} G(z)| \leqslant |D_{\alpha_1} \cdots D_{\alpha_k} F(z)|, \quad \text{for } |z| \geqslant 1. \tag{14}$$

Inequality (14) is clearly equivalent to

$$|D_{\alpha_1} \cdots D_{\alpha_k} Q(z)-n(n-1)\cdots(n-k+1)\bar{\beta}| \leqslant |D_{\alpha_1} \cdots D_{\alpha_k} P(z)-\beta n(n-1) \cdots (n-k+1)\,\alpha_1\alpha_2 \cdots \alpha_k z^{n-k}| \tag{15}$$

for $|z| \geqslant 1$. If $P(z)$ is a polynomial of degree n, then for $|\alpha| \geqslant 1$,

$$\begin{aligned}|D_\alpha p(z)| &= |np(z)+(\alpha-z)\,p'(z)| \\ &\leqslant |np(z)-zp'(z)|+|\alpha|\,|p'(z)| \\ &\leqslant |\alpha|\,(|np(z)-zp'(z)|+|p'(z)|),\end{aligned}$$

which implies for $|z|=1$ and $|\alpha| \geqslant 1$ that [2, Lemma 2]

$$|D_\alpha p(z)| \leqslant |\alpha|\, n \operatorname*{Max}_{|z|=1} |p(z)|.$$

If we apply the above result repeatedly to the polynomial $P(z)$, we get for $|z|=1$,

$$|D_{\alpha_1} \cdots D_{\alpha_k} P(z)| \leqslant n(n-1) \cdots (n-k+1)\, |\alpha_1 \cdots \alpha_k|,$$

from which it follows by the maximum modulus principle that for $|z| \geqslant 1$,

$$|D_{\alpha_1} \cdots D_{\alpha_k} P(z)| \leqslant n(n-1) \cdots (n-k+1)\, |\alpha_1 \alpha_2 \cdots \alpha_k|\, |z|^{n-k}. \quad (16)$$

In view of (16) we can choose an argument of β in (15) such that for $|z| \geqslant 1$,

$$|D_{\alpha_1} \cdots D_{\alpha_k} Q(z)| - n(n-1) \cdots (n-k+1)\, |\beta|$$
$$\leqslant |\beta|\, n(n-1) \cdots (n-k+1)\, |\alpha_1 \cdots \alpha_k|\, |z|^{n-k} - |D_{\alpha_1} \cdots D_{\alpha_k} P(z)|. \quad (17)$$

Now letting $|\beta| \to 1$ in (17), the lemma follows.

For the proof of Theorem 2 we need the following lemma, the proof of which is omitted altogether because it follows along the same lines as that of Lemma 2.

LEMMA 3. *If $P(z)$ is a polynomial of degree n such that* $\operatorname{Max}_{|z|=1} |P(z)| = 1$ *and $\alpha_1, \alpha_2, ..., \alpha_k$, $k \leqslant n-1$, are complex numbers with $|\alpha_i| \leqslant 1$ for all $i = 1, 2, ..., k$, then for $|z| \leqslant 1$,*

$$|D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z)| + |D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} Q(z)|$$
$$\leqslant n(n-1) \cdots (n-k+1)\{|\alpha_1 \alpha_2 \cdots \alpha_k|\, |z|^{n-k} + 1\}, \quad (18)$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. Since the polynomial $P(z)$ has all its zeros in $|z| \geqslant 1$, therefore, for every complex number β such that $|\beta| > 1$, the polynomial $P(z) - \beta Q(z)$, where $Q(z) = z^n \overline{P(1/\bar{z})}$, has all its zeros in $|z| \leqslant 1$. So that if $r > 1$, then the polynomial $P(rz) - \beta Q(rz)$ has all its zeros in $|z| \leqslant 1/r < 1$. It then follows by Lemma 1 that if $\alpha_1, \alpha_2, ..., \alpha_k$ are complex numbers such that $|\alpha_i| \geqslant 1$, $1 \leqslant i \leqslant k$, the polynomial

$$D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k}(P(rz) - \beta Q(rz))$$

has all its zeros in $|z| < 1$, which implies that all the zeros of

$$D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(rz) - \beta\, D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} Q(rz)$$

lie in $|z|<1$ for every β with $|\beta|>1$. This clearly implies that for $|z|\geqslant 1$

$$|D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(rz)| \leqslant |D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} Q(rz)|. \tag{19}$$

Letting $r \to 1$ in (19) and using continuity we obtain for $|z|\geqslant 1$,

$$|D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z)| \leqslant |D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} Q(z)| \tag{20}$$

and the proof of the Theorem 1 follows on combining (20) with Lemma 2.

Proof of Theorem 2. Since $P(z)$ is a self-inversive polynomial, we have

$$P(z) \equiv Q(z) = z^n \overline{P(1/\bar{z})}.$$

Therefore, for all complex numbers $\alpha_1, \alpha_2, ..., \alpha_k$, it follows that

$$D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z) = D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} Q(z). \tag{21}$$

Using (21) in the conclusion of Lemma 2, we obtain

$$\begin{aligned} &2\,|D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z)| \\ &\qquad \leqslant n(n-1)\cdots(n-k+1)\{|\alpha_1\alpha_2 \cdots \alpha_k|\,|z|^{n-k}+1\} \end{aligned} \tag{22}$$

for $|z|\geqslant 1$ where $|\alpha_i|\geqslant 1$ for all $i=1, 2, ..., k$.

Next using (21) in the conclusion of Lemma 3, we get

$$\begin{aligned} &2\,|D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z)| \\ &\qquad \leqslant n(n-1)\cdots(n-k+1)\{|\alpha_1\alpha_2 \cdots \alpha_k|\,|z|^{n-k}+1\} \end{aligned} \tag{23}$$

for $|z|\leqslant 1$ where $|\alpha_i|\leqslant 1$ for all $i=1, 2, ..., k$.

(22) and (23) are equivalent to the assertions of Theorem 2 and this completes the proof.

Proof of Theorem 3. If $|\beta|=1$, then the result follows from the Lemma 2 of [2]. Hence we suppose that $|\beta|>1$. If $Q(z)=z^n \overline{P(1/\bar{z})}$, then

$$Q'(z) = nz^{n-1}\overline{P(1/\bar{z})} - z^{n-2}\overline{P'(1/\bar{z})},$$

so that for points $z=e^{i\theta}$, $0\leqslant\theta<2\pi$, we have

$$\begin{aligned} |D_\alpha Q(e^{i\theta})| &= |nQ(e^{i\theta}) + (\alpha - e^{i\theta})\, Q'(e^{i\theta})| \\ &= |ne^{in\theta}\,\overline{P(e^{i\theta})} + (\alpha - e^{i\theta})(ne^{i(n-1)\theta}\,\overline{P(e^{i\theta})} - e^{i(n-2)\theta}\,\overline{P'(e^{i\theta})})| \\ &= |n\alpha e^{i(n-1)\theta}\,\overline{P(e^{i\theta})} - (\alpha - e^{i\theta})\, e^{i(n-2)\theta}\,\overline{P'(e^{i\theta})}| \\ &= |n\bar{\alpha}P(e^{i\theta}) - (\bar{\alpha}e^{i\theta} - 1)\, P'(e^{i\theta})|. \end{aligned}$$

This shows that

$$|D_\alpha Q(z)| = |n\bar{\alpha}P(z) + (1-\bar{\alpha}z)\,P'(z)|, \quad \text{for } |z|=1. \tag{24}$$

Since $P(z)$ has all its zeros in $|z| \geqslant k \geqslant 1$, therefore, the polynomial $G(z)=P(kz)$ has all its zeros in $|z| \geqslant 1$. Hence if $H(z)=z^n\,\overline{G(1/\bar{z})}$, then it follows from (20) that

$$|D_\beta\, G(z)| \leqslant |D_\beta H(z)|, \quad \text{for } |z|=1 \text{ and } |\beta|>1.$$

This gives with the help of (24) that

$$\begin{aligned}|D_\beta G(z)| &\leqslant |n\bar{\beta}\,G(z) + (1-\bar{\beta}z)\,G'(z)|\\ &= |\beta|\;|nG(z) + ((1/\bar{\beta})-z)\,G'(z)|, \quad \text{for } |z|=1.\end{aligned} \tag{25}$$

Since $|\beta|>1$, it follows by Lemma 1 that all the zeros of

$$D_{1/\bar{\beta}}\,G(z) = nG(z) + ((1/\bar{\beta})-z)\,G'(z)$$

lie in $|z| \geqslant 1$. Hence by the maximum modulus principle, the inequality (25) holds for $|z| \leqslant 1$ also. Replacing $G(z)$ by $P(kz)$, we obtain

$$\begin{aligned}&|nP(kz) + (\beta - z)\,kP'(kz)|\\ &\quad \leqslant |n\bar{\beta}P(kz) + (1-\bar{\beta}z)\,kP'(kz)|, \qquad \text{for} \quad |z| \leqslant 1.\end{aligned}$$

Taking in particular $z=e^{i\theta}/k$, $0 \leqslant \theta < 2\pi$, $k \geqslant 1$, we get

$$|nP(e^{i\theta}) + (\beta k - e^{i\theta})\,P'(e^{i\theta})| \leqslant |n\bar{\beta}\,P(e^{i\theta}) + (k-\bar{\beta}e^{i\theta})\,P'(e^{i\theta})|.$$

This implies

$$|nP(z) + (\beta k - z)\,P'(z)| \leqslant |n\bar{\beta}\,P(z) + (k-\bar{\beta}z)\,P'(z)|, \qquad \text{for} \quad |z|=1,$$

which gives with the help of (24) that

$$|D_{\beta k}\,P(z)| \leqslant k\,|D_{\beta/k}\,Q(z)|, \qquad \text{for} \quad |z|=1. \tag{26}$$

Now from Lemmas 2 and 3, it easily follows that for every complex number δ we have

$$|D_\delta P(z)| + |D_\delta Q(z)| \leqslant n(1+|\delta|), \qquad \text{for} \quad |z|=1.$$

We take in particular $\delta=\beta/k$ and from (26) we get

$$\begin{aligned}(1+k)\,|D_\beta P(z)| &= |D_{\beta k}P(z)+k\,D_{\beta/k}P(z)|\\ &\leqslant |D_{\beta k}P(z)|+k\,|D_{\beta/k}P(z)|\\ &\leqslant k\{|D_{\beta/k}Q(z)|+|D_{\beta/k}P(z)|\}\\ &\leqslant kn(1+|\beta|/k)\\ &= n(k+|\beta|), \qquad \text{for} \quad |z|=1,\end{aligned}$$

which immediately gives (11) and Theorem 3 is proved.

Remark 2. If $P(z)=a_0+a_1z+\cdots+a_nz^n$ is a polynomial of degree n and $Q(z)=z^n\,\overline{P(1/\bar{z})}$, then by Lemma 3, with $k=1$, we get

$$|D_\alpha P(z)|_{z=0}+|D_\alpha Q(z)|_{z=0}\leqslant n\,\underset{|z|=1}{\text{Max}}\,|P(z)|$$

for every α with $|\alpha|\leqslant 1$. This implies

$$|nP(0)+\alpha P'(0)|+|nQ(0)+\alpha Q'(0)|\leqslant n\,\underset{|z|=1}{\text{Max}}\,|P(z)|.$$

Equivalently,

$$|na_0+\alpha a_1|+|na_n+\bar{\alpha}a_{n-1}|<n\,\underset{|z|=1}{\text{Max}}\,|P(z)| \tag{27}$$

for every α with $|\alpha|\leqslant 1$. For $\alpha=0$, (27) reduces to a result due to C. Visser [11].

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